ON THE LENGTHS OF QUOTIENTS OF IDEALS AND DEPTHS OF FIBER CONES

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Abstract. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal of R and J its minimal reduction. We study the depths of $F(I)$ under certain depth assumptions on $G(I)$ and length condition on quotients of powers of I and J, namely $\sum_{n\geq 0} \lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n)$ and $\sum_{n\geq 0} \lambda(\mathfrak{m}I^{n+1} \cap J/\mathfrak{m}JI^n)$.

1. Introduction

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$ with infinite residue field and I an $\mathfrak{m}$-primary ideal of R. The fiber cone of I, denoted by $F(I) := \oplus_{n\geq 0} I^n/\mathfrak{m}I^n$, the associated graded ring of I, denoted by $G(I) := \oplus_{n\geq 0} I^n/I^{n+1}$ and the Rees algebra $\mathcal{R}(I) := \oplus_{n\geq 0} I^n t^n \subset R[t]$ are together known as blowup algebras related to I. Many authors have studied the relationship between properties of the ideal and its blowup algebras. Northcott and Rees introduced the concept of a reduction of an ideal to study various properties of the ideal and its blowup algebras. An ideal $J \subseteq I$ is said to be a reduction of I with respect to an R-module M if $I^{n+1}M = JI^nM$ for some $n \geq 0$. The integer $r_J^M(I) = \min\{n \mid I^{n+1}M = JI^nM\}$ is called the M-reduction number of I with respect to J. If $M = R$, then J is said to be a reduction of I and the integer $r_J(I) = r_J^R(I)$ is known as the reduction number of I with respect to J. A reduction is said to be a minimal reduction if it is minimal with respect to inclusion. It is known that if the residue field of R is infinite, then all minimal reductions are minimally generated by $\ell(I)$ number of elements, where $\ell(I) := \dim F(I)$ is the analytic spread of I.

The relation between the lengths of quotients of ideals and depths of blowup algebras has been a subject of several papers. Let $\lambda(-)$ denote the length function. It has been shown by many authors that the two integers

$$\Delta(I, J) = \sum_{p\geq 1} \lambda\left(\frac{I^{p+1} \cap J}{I^pJ}\right), \qquad \Lambda(I, J) = \sum_{p\geq 0} \lambda\left(\frac{I^{p+1}}{I^pJ}\right)$$

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controls the depth of the associated graded ring. Valabrega and Valla proved that $I^n \cap J = JI^{n-1}$ for all $n \geq 1$ if and only if $G(I)$ is Cohen-Macaulay, where J is a minimal reduction of I, [VV]. The Valabrega-Valla condition can be rephrased as $\sum_{n\geq 1} \lambda(I^n \cap J/JI^{n-1}) = 0$. Guerrieri studied ideals satisfying $\sum_{n\geq 1} \lambda(I^n \cap J/JI^{n-1}) = 1$ and showed that in this case $\operatorname{depth} G(I) = d-1$, [Gu1]. She also proved that if $\lambda(I^2 \cap J/JI) = 2$ and $I^k \cap J = JI^{k-1}$ for all $k \geq 3$, then $\operatorname{depth} G(I) \geq d-2$, [Gu2].

Huckaba and Marley proved that $e_1(I) \leq \Lambda(I, J)$ and if the equality holds, then $\operatorname{depth} G(I) \geq d-1$, [HM]. Wang showed that if $\Lambda(I, J) - e_1(I) = 1$, then $\operatorname{depth} G(I) \geq d-2$. As a consequence he deduced that if $\Delta(I, J) = 2$, then $\operatorname{depth} G(I) \geq d-2$, [W]. Guerrieri and Rossi proved that if $\lambda(I^2 \cap J/JI) = 3$ and $J \cap I^n = JI^{n-1}$ for all $n \geq 3$, then $\operatorname{depth} G(I) \geq d-3$, assuming that R/I is Gorenstein and $d \geq 4$, [GR].

In the case of fiber cone, the similar relations have not been investigated well. Cortadellas and Zarzuela proved that if $G(I)$ is Cohen-Macaulay, then $F(I)$ is Cohen-Macaulay if and only if $\sum_{n\geq 2} \lambda(\mathfrak{m}I^n \cap J/\mathfrak{m}I^{n-1}J) = 0$, [CZ]. It is known that $r(I) \leq 1$ implies that the fiber cone is Cohen-Macaulay, [S]. The relation between $\mathfrak{m}$-reduction number and the depth of fiber cone is not that strong. The fiber cone need not even have depth $d-1$ when $r_J^{\mathfrak{m}}(I) = 1$, cf. Example 5.1.

In this article we study the depths of fiber cones of ideals satisfying the properties $\sum_{n\geq 1} \lambda(\mathfrak{m}I^{n+1} \cap J/\mathfrak{m}JI^n) \leq 1$ and $\sum_{n\geq 0} \lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) \leq 2$ under some depth assumptions on depth of $G(I)$. The paper is organized in the following manner. In Section 2, we present some preliminary lemmas needed for the proof of main theorems. In Section 3, we prove:

Theorem 3.1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $J \subseteq I$ a minimal reduction of I such that $mI^n \cap J = \mathfrak{m}I^{n-1}J$ for all $n \geq 1$. If $\operatorname{depth}(G(I)) \geq d-t$, then $\operatorname{depth}(F(I)) \geq d-t+1$ for $1 \leq t \leq d$.

Theorem 3.3. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $J \subseteq I$ a minimal reduction of I such that $\sum_{k\geq 2} \lambda((\mathfrak{m}I^k \cap J)/\mathfrak{m}I^{k-1}J) = 1$. If $\operatorname{depth}(G(I)) \geq d-t$, then $\operatorname{depth}(F(I)) \geq d-t$ for $1 \leq t \leq d-1$.

In Section 4, we prove:

Theorem 4.1, 4.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field. Let I be any $\mathfrak{m}$-primary ideal of R and $J \subseteq I$ a minimal

reduction of I. Suppose $\sum_{n\geq 0}\lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) = 1$ or 2. If $\operatorname{depth}(G(I)) \geq d-t$, then $\operatorname{depth}(F(I)) \geq d-t+1$, for $2 \leq t \leq d$.

We also study the Cohen-Macaulay property of fiber cones in these cases. In the last section we present some examples to support our results. The computations have been performed in the Computational Commutative Algebra software, CoCoA [Co].

2. Preliminaries

In this section we prove some technical lemmas which are required in the proof of main theorems. Throughout this paper $(R,\mathfrak{m})$ denotes a Cohen-Macaulay local ring with infinite residue field, I an $\mathfrak{m}-$primary ideal and J its minimal reduction. First, we recall some results from the literature that we need.

Theorem 2.1. [BH, Theorem 1.1.7] *Let R be a ring, M an $R-$module,$x_1,\ldots,x_n$ an $M-$regular sequence, and $I=(x_1,\ldots,x_n)$. Let $X_1,\ldots,X_n$ be indeterminates over R. If $F\in M[X_1,\ldots,X_n]$ is homogeneous of total degree d and $F(x_1,\ldots,x_n)\in I^{d+1}M$, then the coefficients of F are in IM.*

Lemma 2.2. [JV1, Lemma 5.2] *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d>0$ with infinite residue field. Let I be an $\mathfrak{m}-$primary ideal and J a minimal reduction of I. Let $\{x_1,\ldots,x_d\}$ be a minimal generating set for J such that for some index i, $1\leq i\leq d$ $\mathfrak{m}I^n\cap(x_1,\ldots,\hat{x_i},\ldots,x_d)\subseteq \mathfrak{m}I^{n-1}J$ for all n, $1\leq n\leq k$ for some integer k. Then for all $1\leq n\leq k$,*

$$\mathfrak{m}I^n\cap(x_1,\ldots,\hat{x_i},\ldots,x_d)=\mathfrak{m}I^{n-1}(x_1,\ldots,\hat{x_i},\ldots,x_d).$$

The following lemma, which is very useful in detecting positive depth property of fiber cones, is known as the "Sally machine for fiber cones".

Lemma 2.3 (Lemma 2.7,[JV1])**.** *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal in R and $x\in I$ such that x^* is superficial in $G(I)$ and x^o is superficial in $F(I)$. If $\operatorname{depth}(F(I/(x)))\geq 1$, then x^o is regular in $F(I)$.*

For the rest of the section, let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d>0$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $J\subseteq I$ a minimal reduction of I such that for some $k\geq 2$, $\mathfrak{m}I^n\cap J=\mathfrak{m}JI^{n-1}$ for $1\leq n<k$.

Lemma 2.4. *Let $\{x_1,\ldots,x_d\}$ be a minimal generating set for J such that $\mathfrak{m}I^n\cap(x_{i_1},\ldots,x_{i_r})\subseteq \mathfrak{m}I^{n-1}J$ for all $n\leq k$ and for some $1\leq i_1<\cdots<i_r\leq d$, where $1\leq r<d$. Then $\mathfrak{m}I^n\cap(x_{i_1},\ldots,x_{i_r})=\mathfrak{m}I^{n-1}(x_{i_1},\ldots,x_{i_r})$ for all $n\leq k$.*

Proof. Without loss of generality we may assume $i_1 = 1, \dots, i_r = r$. The Lemma 5.2 in [JV1] proves the case $r = d-1$. Assume $1 \le r \le d-2$. We proceed by induction on k. Let $k = 2$. Since $\bar{x}_1, \dots, \bar{x}_r$ is a part of an $R/\mathfrak{m}$-basis for $I/\mathfrak{m}I$, $\mathfrak{m}I \cap (x_1, \dots, x_r) = \mathfrak{m}(x_1, \dots, x_r)$. Let $y = a_1x_1 + \cdots + a_rx_r \in \mathfrak{m}I^2 \cap (x_1, \dots, x_r)$. By hypothesis $y = b_1x_1 + \cdots + b_dx_d$, where $b_j \in \mathfrak{m}I$ for all $i = 1, \dots, d$. Therefore $(a_1 - b_1)x_1 + \cdots + (a_r - b_r)x_r = b_{r+1}x_{r+1} + \cdots + b_dx_d$. Since $\{x_1, \dots, x_d\}$ is a regular sequence, $b_{r+1}, \dots, b_d \in (x_1, \dots, x_r) \subseteq J$. $b_{r+1}, \dots, b_d \in \mathfrak{m}I \cap J = \mathfrak{m}J$. Therefore $(a_1 - b_1)x_1 + \cdots + (a_r - b_r)x_r \in \mathfrak{m}J^2$. By the Theorem 2.1, $a_1 - b_1, \dots, a_r - b_r \in \mathfrak{m}J$. This implies $a_1, \cdots, a_r \in \mathfrak{m}I$. Hence $y \in \mathfrak{m}I(x_1, \dots, x_r)$.

Assume $k \ge 3$. By induction hypothesis $\mathfrak{m}I^n \cap (x_1, \dots, x_r) = \mathfrak{m}I^{n-1}(x_1, \dots, x_r)$ for all $n \le k-1$. Since $\mathfrak{m}I^k \cap (x_1, \dots, x_r) = \mathfrak{m}I^{k-1}(x_1, \dots, x_r) + [\mathfrak{m}I^{k-1}(x_{r+1}, \dots, x_d) \cap (x_1, \dots, x_r)]$ it is enough to prove:

Claim: $\mathfrak{m}I^t(x_{r+1}, \dots, x_d)^{n-t} \cap (x_1, \dots, x_r) \subseteq \mathfrak{m}I^{n-1}(x_1, \dots, x_r)$ for all integers t, n such that $0 \le t < k$ and $t < n$.

We prove the claim by induction on t. Suppose $t = 0$. We need to prove that $\mathfrak{m}(x_{r+1}, \dots, x_d)^n \cap (x_1, \dots, x_r) \subseteq \mathfrak{m}J^{n-1}(x_1, \dots, x_r)$ for all $n, r \ge 1$. We prove this statement by induction on n. Since $(x_1, \dots, x_r)$ is a regular sequence, the case $n = 1$ is obvious. Assume that $n \ge 2$ and that the statement is true for all $h < n$. Let $s \in \mathfrak{m}(x_{r+1}, \dots, x_d)^n \cap (x_1, \dots, x_r)$. By induction hypothesis

$$s \in \mathfrak{m}J^{n-2}(x_1, \dots, x_r) = \mathfrak{m}(x_1, \dots, x_r)^{n-1} + \cdots + \mathfrak{m}(x_1, \dots, x_r)(x_{r+1}, \dots, x_d)^{n-2} \subseteq \mathfrak{m}J^{n-1}.$$

Thus we can write

$$\sum_{|\sigma| = n-1} k_\sigma x_1^{\sigma_1} \cdots x_r^{\sigma_r} + \cdots + \sum_{|\rho| = n-2} k_\rho x_r x_{r+1}^{\rho_{r+1}} \cdots x_d^{\rho_d} = s = \sum_{|\omega| = n} j_\omega x_{r+1}^{\omega_{r+1}} \cdots x_d^{\omega_d},$$

where $k_\sigma, \dots, k_\rho \in \mathfrak{m}$. Therefore s is a homogeneous polynomial in $x_1, \dots, x_d$ of degree $n-1$ with coefficients from $\mathfrak{m}$ such that $s \in \mathfrak{m}J^n$ and $x_1, \dots, x_d$ is a regular sequence. By the Theorem 2.1 all the coefficients, $k_\sigma, \dots, k_\rho$ are in $\mathfrak{m}J$. Hence

$$s \in \mathfrak{m}J(x_1, \dots, x_r)^{n-1} + \cdots + \mathfrak{m}J(x_1, \dots, x_r)(x_{r+1}, \dots, x_d)^{n-2} \subseteq \mathfrak{m}J^{n-1}(x_1, \dots, x_r).$$

This proves case $t = 0$. Assume $t \ge 1$. Let $s \in \mathfrak{m}I^t(x_{r+1}, \dots, x_d)^{n-t} \cap (x_1, \dots, x_r)$. Then by what we have shown in proving the case $t = 0$,

$$\mathfrak{m}J^{n-t-1}(x_1, \dots, x_r) = \mathfrak{m}(x_1, \dots, x_r)^{n-t} + \cdots + \mathfrak{m}(x_1, \dots, x_r)(x_{r+1}, \dots, x_d)^{n-t-1}.$$

It is possible to write

$$\sum_{|\sigma|=n-t} k_\sigma x_1^{\sigma_1}\cdots x_r^{\sigma_r} + \cdots + \sum_{|\rho|=n-t-1} k_\rho x_r x_{r+1}^{\rho_{r+1}}\cdots x_d^{\rho_d} = s = \sum_{|\omega|=n-t} j_\omega x_{r+1}^{\omega_{r+1}}\cdots x_d^{\omega_d},$$

where $k_\sigma,\ldots,k_p \in \mathfrak{m}$ and all $j_\omega \in \mathfrak{m}I^t$. Again using the Theorem 2.1, we get $j_\omega \in (x_1,\ldots,x_r)$. Hence $j_\omega \in \mathfrak{m}I^t \cap J$ for all ω. Since $t < k$, by hypothesis we obtain all $j_\omega \in \mathfrak{m}I^{t-1}J$. Therefore

$$s \in \mathfrak{m}I^{t-1}J(x_{r+1},\ldots,x_d)^{n-t} = \mathfrak{m}I^{t-1}(x_1,\ldots,x_r)(x_{r+1},\ldots,x_d)^{n-t} + \mathfrak{m}I^{t-1}(x_{r+1},\ldots,x_d)^{n-t+1}.$$

In particular we have shown that

$$\begin{aligned}\mathfrak{m}I^t(x_{r+1},\ldots,x_d)^{n-t}\cap(x_1,\ldots,x_r) \;\subseteq\; & \mathfrak{m}I^{t-1}(x_1,\ldots,x_r)(x_{r+1},\ldots,x_d)^{n-t}\\ & +\mathfrak{m}I^{t-1}(x_{r+1},\ldots,x_d)^{n-t+1}\cap(x_1,\ldots,x_r).\end{aligned}$$

By induction hypothesis, $\mathfrak{m}I^{t-1}(x_{r+1},\ldots,x_d)^{n-t+1}\cap(x_1,\ldots,x_r) \subseteq \mathfrak{m}I^{n-1}(x_1,\ldots,x_r)$. Therefore

$$\mathfrak{m}I^t(x_{r+1},\ldots,x_d)^{n-t}\cap(x_1,\ldots,x_r) \subseteq \mathfrak{m}I^{n-1}(x_1,\ldots,x_r).$$

This completes the proof of the claim and hence the lemma. □

Lemma 2.5. *Let $y_1,\ldots,y_r$ be elements in $J-\mathfrak{m}J$ such that the sets $\{y_i,y_j\}$ $1\le i<j\le r$ are part of minimal generating sets for J. Then for each $0\le t\le k-1$, we have*

$$(\mathfrak{m}I^t+(y_1))\cap\cdots\cap(\mathfrak{m}I^t+(y_r)) \subseteq \mathfrak{m}I^t+(y_1\cdots y_r).$$

Proof. We prove by induction on r. Assume $r=2$. Let $0\le t\le k-1$. Let $w\in(\mathfrak{m}I^t+(y_1))\cap(\mathfrak{m}I^t+(y_2))$. Then $i_1+w_1y_1 = w = i_2+w_2y_2$ for some $i_1,i_2\in\mathfrak{m}I^t$ and $w_1,w_2\in R$. Therefore $w_1y_1-w_2y_2\in\mathfrak{m}I^t\cap(y_1,y_2)=\mathfrak{m}I^{t-1}(y_1,y_2)$, by Lemma 2.4. This implies that $w_1y_1\in\mathfrak{m}I^{t-1}y_1+(y_2)$. From hypothesis $\{y_1,y_2\}$ is a regular sequence we get $w_1\in\mathfrak{m}I^{t-1}+(y_2)$. Hence $w\in\mathfrak{m}I^t+(y_1y_2)$.

Assume $r\ge 3$ and that the statement is true for each integer strictly less than r. Let $w\in(\mathfrak{m}I^t+(y_1))\cap\cdots\cap(\mathfrak{m}I^t+(y_r))$. Then $w=i_r+w_ry_r$ with $i_r\in\mathfrak{m}I^t$. Thus $w_ry_r\in\cap_{i=1}^{r-1}(\mathfrak{m}I^t+(y_i))$. By induction hypothesis $w_ry_r\in\mathfrak{m}I^t+(y_1\cdots y_{r-1})$. Therefore $w_ry_r = i+\alpha y_1\ldots y_{r-1}$ for some $i\in\mathfrak{m}I^t$ and $\alpha\in R$. This gives $w_ry_r-\alpha y_1\cdots y_{r-1}\in\mathfrak{m}I^t\cap(y_r,y_j)$ for all $j=1,\ldots,r-1$. By the Lemma 2.4, $w_ry_r-\alpha y_1\cdots y_{r-1}\in\mathfrak{m}I^{t-1}(y_r,y_j)$ for all $j=1,\ldots,r-1$. If $w_ry_r-\alpha y_1\cdots y_{r-1}=ay_r+by_j$ for some $a,b\in\mathfrak{m}I^{t-1}$, then for all $1\le j\le r-1$, $(w_r-a)y_r\in(y_j)$ so that $(w_r-a)\in(y_j)$. Thus $w_r\in\cap_{j=1}^{r-1}(\mathfrak{m}I^{t-1}+(y_j))$. By induction hypothesis $w_r\in\mathfrak{m}I^{t-1}+(y_1\ldots y_{r-1})$. Hence $w\in\mathfrak{m}I^t+(y_1,\ldots,y_r)$. □

Lemma 2.6. *Let $y_1, \dots, y_r$ be elements in $J - \mathfrak{m}J$ such that the sets $\{y_i, y_j\}$ are part of minimal generating sets for J for $1 \leq i < j \leq r$. Suppose that $y_1, \dots, y_r$ satisfy the following equalities: $\mathfrak{m}I^k \cap (y_1) = \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1y_i)$ for all $i = 2, \dots, r$. Then $\mathfrak{m}I^k \cap (y_1) = \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_r)$.*

Proof. We proceed by induction on r. If $r = 2$, then there is nothing to prove. Assume $r \geq 3$. By induction hypothesis, we have the following two equalities:

$$\mathfrak{m}I^k \cap (y_1) = \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_{r-1}) \text{ and } \mathfrak{m}I^k \cap (y_1) = \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1y_r).$$

Therefore $\mathfrak{m}I^k \cap (y_1y_r) \subseteq \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_{r-1})$. Suppose $\beta y_1y_r \in \mathfrak{m}I^k \cap (y_1y_r)$. Then $\beta y_1y_r = iy_1 + \alpha y_1 \cdots y_{r-1}$ for some $i \in \mathfrak{m}I^{k-1}$ and $\alpha \in R$. As y_1 is regular, $\beta y_r = i + \alpha y_2 \cdots y_{r-1}$. This implies $\beta y_r \in \cap_{j=2}^{r-1}[\mathfrak{m}I^{k-1} + (y_j)]$. By the proof of Lemma 2.5, we get $\beta \in \cap_{j=2}^{r-1}(\mathfrak{m}I^{k-2} + (y_j))$ so that by Lemma 2.5 we get $\beta \in \mathfrak{m}I^{k-2}(y_2 \cdots y_{r-1})$. Therefore $\beta y_1y_r \in (\mathfrak{m}I^{k-2} + (y_2 \cdots y_{r-1}))(y_1y_r) \subseteq \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_r)$ which in turn gives $\mathfrak{m}I^k \cap (y_1) \subseteq \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_r)$. Hence $\mathfrak{m}I^k \cap (y_1) = \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_r)$. $\square$

Lemma 2.7. *Let $d \geq 2$. If $\lambda((\mathfrak{m}I^k \cap J)/\mathfrak{m}I^{k-1}J) \neq 0$ for some $k \geq 2$ and $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$ for all n, $1 \leq n < k$, then there exists an element $x \in J - \mathfrak{m}J$ such that*

(1) *$x^* \in G(I)$ is superficial in $G(I)$*
(2) *$x^o \in F(I)$ is superficial in $F(I)$ and*
(3) *$\lambda((\mathfrak{m}I^k \cap J)/(\mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x))) \neq 0$.*

Proof. Since it is possible to choose a minimal generating set $\{x_1, \dots, x_d\}$ for J such that $x_1^*, \dots, x_d^*$ is a superficial sequence in $G(I)$ and $x_1^o, \dots, x_d^o$ is a superficial sequence in $F(I)$, we show that the result holds for a part of a minimal generating set of J.

We prove the lemma by induction on k. Let $k = 2$. Let $\{x_1 \dots, x_d\}$ be a minimal basis for J. Suppose that

$$\lambda((\mathfrak{m}I^2 \cap J)/(\mathfrak{m}IJ + \mathfrak{m}I^2 \cap (x_i))) = 0 = \lambda((\mathfrak{m}I^2 \cap J)/(\mathfrak{m}IJ + \mathfrak{m}I^2 \cap (x_j)))$$

for some $i \neq j$, $1 \leq i,j \leq d$. Then $\mathfrak{m}I^2 \cap (x_j) \subseteq \mathfrak{m}IJ + \mathfrak{m}I^2 \cap (x_i)$. Let $ax_j \in \mathfrak{m}I^2 \cap (x_j)$. Then $ax_j = a_1x_1 + \cdots + a_dx_d + bx_i$ for some $a_1, \dots, a_d \in \mathfrak{m}I$. This implies $(a - a_j)x_j - (b + a_i)x_i = a_1x_1 + \cdots + \widehat{a_ix_i} + \cdots + \widehat{a_jx_j} + \cdots + a_dx_d$. Since $\{x_1, \dots, x_d\}$ is a regular sequence, $a - a_j, b + a_i \in I$ and hence $a, b \in I$. Again from the same argument we get $a_l \in (x_i, x_j) \subseteq J$ for all $l \neq i, j$. Since $\mathfrak{m}I \cap J = \mathfrak{m}J$, all these coefficients are in $\mathfrak{m}J$. Thus $(a - a_j)x_j - (b + a_i)x_i$ is a homogeneous polynomial of degree 1 in x_i, x_j which belongs to $\mathfrak{m}J^2$. By the Theorem 2.1 we get $a - a_j, b + a_i \in \mathfrak{m}J$. But $a_i, a_j \in \mathfrak{m}I$.

Hence $a, b \in \mathfrak{m}I$. So $ax_j \in \mathfrak{m}Ix_j$. We proved $\mathfrak{m}I^2 \cap (x_j) = \mathfrak{m}Ix_j$. However this yields the contradiction

$$0 = \lambda((\mathfrak{m}I^2 \cap J)/(\mathfrak{m}IJ + \mathfrak{m}I^2 \cap (x_j))) = \lambda((\mathfrak{m}I^2 \cap J)/\mathfrak{m}IJ) \neq 0.$$

Hence for each set of minimal generators for J there are at least $d-1$ elements which satisfy our requirement.

Let $k \geq 3$. Let $\{x_1, \dots, x_d\}$ be a minimal generating set for J. Suppose $\lambda((\mathfrak{m}I^k \cap J)/(\mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (w))) = 0$ for all $w \in J - \mathfrak{m}J$. In particular we have that $\lambda((\mathfrak{m}I^k \cap J)/(\mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x_i))) = 0$ for all $i = 1, \dots, d$. This means $\mathfrak{m}I^k \cap J = \mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x_i)$ for all $i = 1, \dots, d$. Fix an integer i such that $2 \leq i \leq d$. Since $\mathfrak{m}I^k \cap (x_1) \subseteq \mathfrak{m}I^k \cap J = \mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x_i)$, it can easily be seen that

$$\mathfrak{m}I^k \cap (x_1) = \mathfrak{m}I^{k-1}(x_1) + [\mathfrak{m}I^{k-1}(x_2, \cdots, \hat{x}_i, \cdots, x_d) + \mathfrak{m}I^k \cap (x_i)] \cap (x_1).$$

If we choose an element $j_2x_2 + \cdots + \widehat{j_ix_i} + \cdots + j_dx_d + r_ix_i \in [\mathfrak{m}I^{k-1}(x_2, \dots, \hat{x}_i, \dots, x_d) + \mathfrak{m}I^k \cap (x_i)] \cap (x_1)$ for some $j_h \in \mathfrak{m}I^{k-1}$ and $h = 2, \dots, \hat{i}, \dots, d$, then it is clear that $j_2x_2 + \cdots + \widehat{j_ix_i} + \cdots + j_dx_d \in (x_1, x_i)$. Since $\{x_1, \dots, x_d\}$ is a regular sequence, $j_h \in \mathfrak{m}I^{k-1} \cap (x_1, \dots, \hat{x}_h, \dots, x_d)$, for all $h = 2, \dots, \hat{i}, \dots, d$. By Lemma 2.4, $\mathfrak{m}I^{k-1} \cap (x_1, \dots, \hat{x}_h, \dots, x_d) = \mathfrak{m}I^{k-2}(x_1, \dots, \hat{x}_h, \dots, x_d)$, for all $h = 2, \dots, \hat{i}, \dots, d$. Therefore $j_h \in \mathfrak{m}I^{k-2}(x_1, \dots, \hat{x_h}, \dots, x_d)$ for all $h = 2, \dots, \hat{i}, \dots, d$ and hence $j_2x_2 + \cdots + \widehat{j_ix_i} + \cdots + j_dx_d \in \mathfrak{m}I^{k-2}J(x_2, \dots, \hat{x}_i, \dots, x_d) \cap (x_1, x_i)$. Using the CLAIM in the Lemma 2.4 with $t = k-2$ and $n = k$ we can conclude that $j_2x_2 + \cdots + \widehat{j_ix_i} + \cdots + j_dx_d \in \mathfrak{m}I^{k-1}(x_1, x_i)$. Hence

$$\begin{aligned}
\mathfrak{m}I^k \cap (x_1) &\subseteq \mathfrak{m}I^{k-1}x_1 + [\mathfrak{m}I^{k-1}(x_1, x_i) + \mathfrak{m}I^k \cap (x_i)] \cap (x_1) \\
&= \mathfrak{m}I^{k-1}x_1 + [\mathfrak{m}I^{k-1}(x_1) + \mathfrak{m}I^k \cap (x_i)] \cap (x_1) \\
&= \mathfrak{m}I^{k-1}x_1 + [\mathfrak{m}I^{k-1}(x_1) + \mathfrak{m}I^k \cap (x_i) \cap (x_1)] = \mathfrak{m}I^{k-1}x_1 + \mathfrak{m}I^k \cap (x_1x_i).
\end{aligned}$$

The other inclusion is obvious. Therefore $\mathfrak{m}I^k \cap (x_1) = \mathfrak{m}I^{k-1}x_1 + \mathfrak{m}I^k \cap (x_1x_i)$ for all $i = 2, \dots, d$. Hence, by Lemma 2.6 it follows that $\mathfrak{m}I^k \cap (x_1) = \mathfrak{m}I^{k-1}x_1 + \mathfrak{m}I^k \cap (x_1 \cdots x_d)$. If $k < d$, then $x_1 \cdots x_d \in \mathfrak{m}I^{k-1}x_1 \subseteq \mathfrak{m}I^k$ so that $\mathfrak{m}I^k \cap (x_1) = \mathfrak{m}I^{k-1}x_1 + \mathfrak{m}I^k \cap (x_1 \cdots x_d) = \mathfrak{m}I^{k-1}x_1$ which yields the contradiction

$$0 = \lambda((\mathfrak{m}I^k \cap J)/(\mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x_1))) = \lambda((\mathfrak{m}I^k \cap J)/\mathfrak{m}I^{k-1}J) \neq 0.$$

Suppose $k \geq d$. Since $J/\mathfrak{m}J$ is a vector space over an infinite field it is possible to find elements $x_h \in \cap_{i=1}^{h-1}[J - (\mathfrak{m}J + (x_i))]$ for $h = d+1, \dots, k+1$ so that $\{x_1, \dots, x_{d-1}, x_{d+1}\}$, $\dots, \{x_1, \dots, x_{d-1}, x_{k+1}\}$ are minimal generating sets for J. Moreover, for any $d+1 \leq h \leq k+1$, by the selection of x_h, $\{\bar{x}_j, \bar{x}_h\} \in J/\mathfrak{m}J$ is $R/\mathfrak{m}$-linearly independent for any $j < h$ and hence form a part of minimal generating set for J. Define $y_1 = x_1, \dots, y_d =$

$x_d,\ y_{d+1} = x_{d+1}, \dots, y_{k+1} = x_{k+1}$. Then $\{y_i, y_j\}$ is a part of minimal generating set of J for $1 \le i, j \le k+1,\ i \ne j$. Also we have $\mathfrak{m}I^k \cap (x_1) = \mathfrak{m}I^{k-1}x_1 + \mathfrak{m}I^k \cap (x_1x_i)$ for all $i = d, \dots, k$. Thus $y_1, \dots, y_{k+1}$ satisfy the hypotheses of Lemma 2.6. Therefore $\mathfrak{m}I^k \cap (y_1) = \mathfrak{m}I^{k-1}y_1 + \mathfrak{m}I^k \cap (y_1 \cdots y_{k+1})$. This implies $\mathfrak{m}I^k \cap (x_1) = \mathfrak{m}I^{k-1}x_1$. This gives the contradiction

$$0 = \lambda((\mathfrak{m}I^k \cap J)/(\mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x_1))) = \lambda((\mathfrak{m}I^k \cap J)/\mathfrak{m}I^{k-1}J) \ne 0.$$

This completes the proof of the lemma. □

3. Ideals with $\sum_{n\ge1} \lambda(\mathfrak{m}I^{n+1} \cap J/\mathfrak{m}JI^n) \le 1$

In this section we study ideals satisfying the property $\sum_{n\ge1} \lambda(\mathfrak{m}I^{n+1} \cap J/\mathfrak{m}JI^n) \le 1$. It is known that if $G(I)$ is Cohen-Macaulay, then $F(I)$ is Cohen-Macaulay if and only if $\mathfrak{m}I^n \cap J = \mathfrak{m}JI^{n-1}$ for all $n \ge 1$, [CZ, Theorem 3.2]. We relax the depth condition on $G(I)$, in the sufficiency part of the result of Cortadellas and Zarzuela. We also give an example to show that if $\operatorname{depth} G(I) = d-1$ and $F(I)$ is Cohen-Macaulay, then the equation $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$ need not hold, cf. Example 5.3.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \ge 1$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and J a minimal reduction such that $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$ for all $n \ge 1$. If $\operatorname{depth} G(I) \ge d-t$, then $\operatorname{depth} F(I) \ge d-t+1$ for all $1 \le t \le d$. In particular, if $\operatorname{depth} G(I) \ge d-1$, then $F(I)$ is Cohen-Macaulay.*

Proof. Suppose $d = 1$. Let $J = (x)$. Then $\mathfrak{m}I^n \cap (x) = \mathfrak{m}I^{n-1}(x)$ for all $n \ge 1$. Hence x^o is regular in $F(I)$.

Suppose $d \ge 2$. Assume $\operatorname{depth} G(I) \ge d-t$. Let $x \in J$ be a minimal generator of J such that x^* is superficial in $G(I)$ and x^o is superficial in $F(I)$. Let "-" denotes modulo (x). If $t = d$, then by induction $\operatorname{depth} F(\bar{I}) \ge 1$. By Lemma 2.3, x^o is regular in $F(I)$, and hence $\operatorname{depth} F(I) \ge 1$. If $t \le d-1$, then x^* is regular in $G(I)$ and so $F(\bar{I}) \cong F(I)/x^oF(I)$. By induction, $\operatorname{depth} F(\bar{I}) \ge d-t \ge 1$. Again by Lemma 2.3, x^o is regular in $F(I)$. Hence $\operatorname{depth} F(I) = \operatorname{depth} F(\bar{I}) + 1 \ge d-t+1$. □

Corollary 3.2. (1) *If $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$ for all $n \ge 1$, then $\operatorname{depth} F(I) > 0$.*
(2) *If $r_J^{\mathfrak{m}}(I) = 1$ and $\operatorname{depth} G(I) \ge d-t$, then $\operatorname{depth} F(I) \ge d-t+1$ for all $1 \le t \le d$.*

Proof. Putting $t = d$ in Theorem 3.1, the statement (1) follows. For (2), note that if $\mathfrak{m}I^n = \mathfrak{m}I^{n-1}J$, then $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$. Now the assertion follows from Theorem 3.1. □

Now we study the ideals satisfying the property $\sum_{n\geq 2}\lambda(\mathfrak{m}I^n\cap J/\mathfrak{m}I^{n-1}J)=1$. We prove that in this case the depth of the fiber cone is at least as much as that of the associated graded ring, except when the associated graded ring is Cohen-Macaulay. We also provide an example which shows that the lower bound on the depth of $F(I)$ is sharp.

Theorem 3.3. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d\geq 2$ with infinite residue field, I be an $\mathfrak{m}$-primary ideal in R and $J\subseteq I$ a minimal reduction of I such that $\sum_{n\geq 1}\lambda((\mathfrak{m}I^{n+1}\cap J)/\mathfrak{m}I^nJ)=1$. If* $\operatorname{depth}(G(I))\geq d-t$, *then* $\operatorname{depth}(F(I))\geq d-t$, *for all $1\leq t\leq d-1$.*

Proof. By hypothesis, there exists an integer $k\geq 2$ such that $\lambda(\mathfrak{m}I^k\cap J/\mathfrak{m}I^{k-1}J)=1$ and $\mathfrak{m}I^n\cap J=\mathfrak{m}I^{n-1}J$ for all $n\neq k$. By Lemma 2.7, there exists $x\in J-\mathfrak{m}J$ such that x^* is superficial in $G(I)$, x^o is superficial in $F(I)$ and $\lambda(\mathfrak{m}I^k\cap J/\mathfrak{m}I^{k-1}J+\mathfrak{m}I^k\cap(x))=1$.

We first prove the case $t=d-1$. We do this by induction on d. Let $d=2$. Since $\lambda(\mathfrak{m}I^k\cap J/\mathfrak{m}I^{k-1}J)=\lambda(\mathfrak{m}I^k\cap J/\mathfrak{m}I^{k-1}J+\mathfrak{m}I^k\cap(x))=1$, it follows that $\mathfrak{m}I^k\cap(x)\subseteq\mathfrak{m}I^{k-1}J$. Also, for all $n\neq k$, $\mathfrak{m}I^n\cap(x)\subseteq\mathfrak{m}I^n\cap J=\mathfrak{m}I^{n-1}J$. Hence we have $\mathfrak{m}I^n\cap(x)\subseteq\mathfrak{m}I^{n-1}J$ for all $n\geq 1$. Since x^* is superficial in $G(I)$ and $\operatorname{depth}G(I)\geq 1$, x^* is regular in $G(I)$. Therefore, by Lemma 2.2, x^o is a regular element in $F(I)$.

Assume $d\geq 3$. Let "-" denotes modulo (x). By repeatedly applying Lemma 2.7, choose $\bar{x}_2,\ldots,\bar{x}_{d-1}\in\bar{J}-\bar{\mathfrak{m}}\bar{J}$ such that $\lambda(\bar{\mathfrak{m}}\bar{I}^k\cap\bar{J}/\bar{\mathfrak{m}}\bar{I}^{k-1}\bar{J}+\bar{\mathfrak{m}}\bar{I}^k\cap(\bar{x}_2,\ldots,\bar{x}_{d-1}))=1$. This implies that

$$\bar{\mathfrak{m}}\bar{I}^{k-1}\bar{J}+\bar{\mathfrak{m}}\bar{I}^k\cap(\bar{x}_2,\ldots,\bar{x}_{d-1})=\bar{\mathfrak{m}}\bar{I}^{k-1}\bar{J}.$$

Lifting the equation back to R, we get

$$\mathfrak{m}I^{k-1}J+(x)=\mathfrak{m}I^{k-1}J+\mathfrak{m}I^k\cap(x,x_2,\ldots,x_{d-1})+(x).$$

Intersecting with $\mathfrak{m}I^k$ we get

$$\mathfrak{m}I^{k-1}J+\mathfrak{m}I^k\cap(x)=\mathfrak{m}I^{k-1}J+\mathfrak{m}I^k\cap(x,x_2,\ldots,x_{d-1})$$

By the choice of x, $\mathfrak{m}I^k\cap(x)\subseteq\mathfrak{m}I^{k-1}J$. Therefore $\mathfrak{m}I^k\cap(x,x_2,\ldots,x_{d-1})\subseteq\mathfrak{m}I^{k-1}J$. For $n\neq k$, this inequality anyway holds because of the hypothesis. Therefore we have, $\mathfrak{m}I^n\cap(x,x_2\ldots,x_{d-1})\subseteq\mathfrak{m}I^{n-1}J$ for all $n\geq 1$. By Lemma 2.2, $\mathfrak{m}I^n\cap(x,x_2,\ldots,x_{d-1})=\mathfrak{m}I^{n-1}(x,x_2,\ldots,x_{d-1})$. Going modulo $(x,x_2,\ldots,x_{d-2})$, we obtain $\bar{\mathfrak{m}}\bar{I}^n\cap(\bar{x}_{d-1})=\bar{\mathfrak{m}}\bar{I}^{n-1}(\bar{x}_{d-1})$ for all $n\geq 1$. This implies that $\bar{x}^o_{d-1}\in F(\bar{I})$ is a regular element, where "-" denotes modulo $(x,x_2,\ldots,x_{d-2})$, i.e., $\operatorname{depth}F(\bar{I})\geq 1$. By repeatedly applying Sally machine for fiber cones, Lemma 2.3, we get $\operatorname{depth}F(I)\geq 1$.

Now assume $d \geq 3,\ 1 \leq t \leq d-2$ and $\operatorname{depth} G(I) \geq d-t$. Let $\{x_1 = x, x_2, \ldots, x_d\}$ be a minimal generating set for J such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(I)$, $x_1^o, \ldots, x_d^o$ is a superficial sequence in $F(I)$ and $\lambda(\mathfrak{m}I^k \cap J/\mathfrak{m}I^{k-1}J + \mathfrak{m}I^k \cap (x_1, \ldots, x_i)) = 1$ for all $1 \leq i \leq d-t-1$. Taking modulo $(x_1, \ldots, x_{d-t-1})$, we get $\lambda(\bar{\mathfrak{m}}\bar{I}^k \cap \bar{J}/\bar{\mathfrak{m}}\bar{I}^{k-1}\bar{J}) = 1$, $\bar{\mathfrak{m}}\bar{I}^n \cap \bar{J} = \bar{\mathfrak{m}}\bar{I}^{n-1}\bar{J}$ for all $n \neq k$, and $\operatorname{depth} G(\bar{I}) \geq 1$. Therefore, by the first part of the proof, we get $\operatorname{depth} F(\bar{I}) \geq 1$ so that $\bar{x}_{d-t}^o$ is regular in $F(\bar{I})$. Since $x_1^*, \ldots, x_{d-t}^*$ is a regular sequence in $G(I)$, $F(\bar{I}) \cong F(I)/(x_1^o, \ldots, x_{d-t-1}^o)$. By repeated application of Lemma 2.3, we obtain that $x_1^o, \ldots, x_{d-t}^o$ is a regular sequence in $F(I)$. Hence $\operatorname{depth} F(I) \geq d-t$. $\square$

Remark 3.4. *By* [CZ, Theorem 3.2] *and Theorem 3.3, if* $\sum_{n\geq 2} \lambda(\mathfrak{m}I^n \cap J/\mathfrak{m}JI^{n-1}) = 1$ *and* $G(I)$ *Cohen-Macaulay, then* $\operatorname{depth} F(I) = d-1$. *This has an interesting consequence, that a necessary condition for* $F(I)$ *to be Cohen-Macaulay in this case is that* $\operatorname{depth} G(I) \leq d-1$. *In Section 5, we have provided examples with* $\sum_{n\geq 2} \lambda(\mathfrak{m}I^n \cap J/\mathfrak{m}JI^{n-1}) = 1$ *and*

(1) $G(I)$ *Cohen-Macaulay, but* $F(I)$ *not Cohen-Macaulay;*
(2) $F(I)$ *Cohen-Macaulay, but* $G(I)$ *not Cohen-Macaulay.*

Corollary 3.5. *Suppose* $\lambda(\mathfrak{m}I^2/\mathfrak{m}IJ) = 1$ *and* $\mathfrak{m}I^3 = \mathfrak{m}I^2J$. *If* $\operatorname{depth}(G(I)) \geq d-t$, *then* $\operatorname{depth}(F(I)) \geq d-t$ *for all* $1 \leq t \leq d-1$.

Proof. Since $(\mathfrak{m}I^2 \cap J)/\mathfrak{m}IJ$ is a submodule of $\mathfrak{m}I^2/\mathfrak{m}IJ$, $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}IJ) \leq 1$ and $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$ for all $n \geq 3$. If $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}IJ) = 0$, then $\mathfrak{m}I^n \cap J = \mathfrak{m}I^{n-1}J$ for all $n \geq 1$. Let $\{x_1, \ldots, x_d\}$ be a minimal generating set for J such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(I)$ and $x_1^o, \ldots, x_d^o$ is a superficial sequence in $F(I)$. Then we get $\mathfrak{m}I^n \cap (x_1, \ldots, x_{d-t}) \subseteq \mathfrak{m}I^{n-1}J$ for all $n \geq 1$. If $\operatorname{depth} G(I) \geq d-t$, then the assertion follows from Lemma 2.4 and Theorem 28 of [CZ].

If $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}IJ) = 1$, the assertion follows from Theorem 3.3. $\square$

We conclude this section by deriving a result analogues to a result by Vasconcelos on the depth of the associated graded rings.

Corollary 3.6. *Suppose* $I^3 = JI^2$ *and* $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}JI) = 1$. *Then*

(a) *If* $\lambda(I^2/JI) = 1$, *then* $\operatorname{depth}(F(I)) \geq d-1$.
(b) *If* $\lambda(I^2/JI) = 2$, *then* $\operatorname{depth}(F(I)) \geq d-2$.

Proof. From the hypothesis we have $\sum_{k\geq 2} \lambda((\mathfrak{m}I^k \cap J)/\mathfrak{m}I^{k-1}J) = 1$. Assume $\lambda(I^2/JI) = 1$ and $I^3 = JI^2$. Then by Corollary 2.3(a) in [Gu2] we have $\operatorname{depth}(G(I)) \geq d-1$. Therefore by the Theorem 3.3 we get $\operatorname{depth}(F(I)) \geq d-1$. This proves (a). Now assume $\lambda(I^2/JI) = 2$ and $I^3 = JI^2$. Then by the Corollary 2.3(b) in [Gu2] we have

$\operatorname{depth}(G(I)) \geq d-2$. Therefore by the Theorem 3.3 we have $\operatorname{depth}(F(I)) \geq d-2$. This proves (b). □

4. Ideals with $\sum_{n\geq 0} \lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) = 1$ or 2

In this section, we study fiber cones of ideals with $\sum_{n\geq 1} \lambda(\mathfrak{m}I^n/\mathfrak{m}I^{n-1}J) = 1$ or 2. In [JV2], it was proved that if $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$ and $\operatorname{depth} G(I) \geq d-2$, then $\operatorname{depth} F(I) \geq d-1$. In the following theorem we generalize this result.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field $R/\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary ideal in R with $J \subseteq I$ a minimal reduction of I such that $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$. If $\operatorname{depth}(G(I)) \geq d-t$, then $\operatorname{depth}(F(I)) \geq d-t+1$ for $2 \leq t \leq d$.*

Proof. We prove the result on d. For $d = 2$, the result follows from Corollary 4.5 in [JV2]. Assume $d \geq 3$. We first prove the result for $t = d$. We show that if $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$, then $\operatorname{depth}(F(I)) \geq 1$. Let $x \in J - \mathfrak{m}J$ be an element such that x^* is superficial in $G(I)$ and x^o is superficial in $F(I)$. Let "-" denotes modulo (x). Since $\mathfrak{m}I \cap (x) = \mathfrak{m}(x)$, $\bar{\mathfrak{m}}\bar{I}/\bar{\mathfrak{m}}\bar{J} \cong \mathfrak{m}I/\mathfrak{m}J + \mathfrak{m}I \cap (x) = \mathfrak{m}I/\mathfrak{m}J$. Then $(\bar{R}, \bar{\mathfrak{m}})$ is a $d-1$ dimensional Cohen-Macaulay local ring and $\lambda(\bar{\mathfrak{m}}\bar{I}/\bar{\mathfrak{m}}\bar{J}) = \lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$. Therefore by induction hypothesis $\operatorname{depth}(F(\bar{I})) \geq 1$. Hence by the Lemma 2.3, $\operatorname{depth}(F(I)) \geq 1$. This proves the case $t = d$.

Now assume $2 \leq t \leq d-1$. Choose $x \in J - \mathfrak{m}J$ such that x^* is superficial in $G(I)$ and x^o is superficial in $F(I)$. Let "-" denotes modulo (x). Then $(\bar{R}, \bar{\mathfrak{m}})$ is a $d-1$ dimensional Cohen-Macaulay local ring with $\lambda(\bar{\mathfrak{m}}\bar{I}/\bar{\mathfrak{m}}\bar{J}) = 1$ and $\operatorname{depth} G(\bar{I}) \geq d-t-1$. Induction hypothesis yields that $\operatorname{depth} F(\bar{I}) \geq d-t$. Since $d-t \geq 1$, by Lemma 2.3, x^o is regular in $F(I)$. Since x^* is a regular element in $G(I)$, $F(\bar{I}) \cong F(I)/x^oF(I)$. Hence $\operatorname{depth} F(I) = \operatorname{depth} F(\bar{I}) + 1 \geq d-t+1$. □

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Let I be any $\mathfrak{m}$-primary and $J \subseteq I$ a minimal reduction of I such that $\sum_{n\geq 0} \lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) = 2$. If $\operatorname{depth}(G(I)) \geq d-t$, then $\operatorname{depth}(F(I)) \geq d-t+1$, for all $2 \leq t \leq d$.*

Proof. First we prove the theorem for $t = d$, i.e., we show that $\operatorname{depth}(F(I)) \geq 1$. We do this by induction on d. Suppose $d = 2$. Since $\sum_{n\geq 0} \lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) = 2$ there are two possible cases, namely,

(i) $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1 = \lambda(\mathfrak{m}I^2/\mathfrak{m}JI)$ and $\mathfrak{m}I^{j+1} = \mathfrak{m}JI^j$ for all $j \geq 2$
(ii) $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 2$ and $\mathfrak{m}I^{j+1} = \mathfrak{m}JI^j$ for all $j \geq 1$.

In the first case, the assertion follows by Corollary 4.5 in [JV2]. Now assume that $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 2$ and $\mathfrak{m}I^{j+1} = \mathfrak{m}JI^j$ for all $j \geq 1$. Let $\{x, y\}$ be a minimal generating set for J such that x^*, y^* is a superficial sequence in $G(I)$ and x^o, y^o is a superficial sequence in $F(I)$. To show x^o is a regular element in $F(I)$ it is enough to prove the claim below.

Claim : $(\mathfrak{m}I^{j+1} : x) = \mathfrak{m}I^j$ for all $j \geq 0$.

We prove the claim by induction on j. Since x is a part of minimal generating set for J and hence I, $j = 0$ case holds. Assume $j \geq 1$ and the induction hypothesis that $(\mathfrak{m}I^j : x) = \mathfrak{m}I^{j-1}$. To show that $(\mathfrak{m}I^{j+1} : x) = \mathfrak{m}I^j$. Consider the following exact sequence for $j \geq 1$,

$$0 \longrightarrow (\mathfrak{m}I^j : x)/(\mathfrak{m}I^j : J) \xrightarrow{\mu_y} (\mathfrak{m}I^{j+1} : x)/\mathfrak{m}I^j \xrightarrow{\mu_x} \mathfrak{m}I^{j+1}/\mathfrak{m}JI^j \longrightarrow \bar{\mathfrak{m}}\bar{I}^{j+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^j \longrightarrow 0,$$

where "-" denotes the modulo (x). The first map μ_y is the multiplication by y and second map μ_x is the multiplication by x. Since $\mathfrak{m}I^{j+1} = \mathfrak{m}JI^j$ for all $j \geq 1$, the last two modules of the above exact sequence are zeros. Hence the first two modules are isomorphic. That is $(\mathfrak{m}I^j : x)/(\mathfrak{m}I^j : J) \cong (\mathfrak{m}I^{j+1} : x)/\mathfrak{m}I^j$. Then $\mathfrak{m}I^{j-1} \subseteq (\mathfrak{m}I^j : J) \subseteq (\mathfrak{m}I^j : x) = \mathfrak{m}I^{j-1}$, where the last equality follows by induction hypothesis. This implies that $(\mathfrak{m}I^j : x) = (\mathfrak{m}I^j : J)$. From the above isomorphism $(\mathfrak{m}I^{j+1} : x) = \mathfrak{m}I^j$ as required. This proves the claim. Therefore x^o is regular in $F(I)$. Hence $\operatorname{depth}(F(I)) \geq 1$.

Assume $d \geq 3$. Let $\{x_1, \ldots, x_d\}$ be a minimal generating set for J such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(I)$ and $x_1^o, \ldots, x_d^o$ is a superficial sequence in $F(I)$. Let "-" denotes modulo (x_1). Then $(\bar{R}, \bar{\mathfrak{m}})$ is a $(d-1)$-dimensional Cohen-Macaulay local ring with

$$0 \neq \sum_{n \geq 0} \lambda(\bar{\mathfrak{m}}\bar{I}^{n+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^n) \leq 2.$$

If $\lambda(\bar{\mathfrak{m}}\bar{I}/\bar{\mathfrak{m}}\bar{J}) = 1$, then the result follows from Theorem 4.1 and Lemma 2.3. If $\lambda(\bar{\mathfrak{m}}\bar{I}^{n+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^n) = 2$, then again the assertion follows by induction hypothesis and Lemma 2.3. This proves the theorem for the case $t = d$.

Suppose $2 \leq t \leq d-1$ and $\operatorname{depth} G(I) \geq d - t$. Let $\{x_1, \ldots, x_d\}$ be a minimal generating set for J such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(I)$ and $x_1^o, \ldots, x_d^o$ is a superficial sequence in $F(I)$. Let "-" denotes the modulo (x_1). Then $(\bar{R}, \bar{\mathfrak{m}})$ is a $(d-1)$-dimensional Cohen-Macaulay local ring with $\operatorname{depth}(G(\bar{I})) \geq d - 1 - t$ and

$$0 \neq \sum_{n \geq 0} \lambda(\bar{\mathfrak{m}}\bar{I}^{n+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^n) \leq 2.$$

If $\sum_{n\geq 0}\lambda(\bar{\mathfrak{m}}\bar{I}^{n+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^n) = 1$, then by Theorem 4.1 we get $\operatorname{depth}(F(\bar{I})) \geq d-t$. If $\sum_{n\geq 0}\lambda(\bar{\mathfrak{m}}\bar{I}^{n+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^n) = 2$, then by induction hypothesis, $\operatorname{depth} F(\bar{I}) \geq d-t$. Since $d-t\geq 1$, by Lemma 2.3 x_1^o is regular in $F(I)$. Moreover, since $\operatorname{depth} G(I) \geq d-t \geq 1$, x_1^* is regular in $F(I)$ so that $F(\bar{I}) \cong F(I)/x_1^o F(I)$. Thus $\operatorname{depth}(F(I)) = \operatorname{depth}(F(\bar{I})) + 1 \geq d-t+1$. □

Corollary 4.3. *If* $\sum\limits_{n\geq 0}\lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) \leq 2$, *then* $\operatorname{depth}(F(I)) \geq 1$.

Proof. From the proofs of Theorem 4.1 and Theorem 4.2 we get that if

$$0 \neq \sum_{n\geq 0}\lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) \leq 2,$$

then $\operatorname{depth}(F(I)) \geq 1$. Now assume $\sum_{n\geq 0}\lambda(\mathfrak{m}I^{n+1}/\mathfrak{m}JI^n) = 0$. This implies that $\mathfrak{m}I = \mathfrak{m}J$. Now the result follows from Lemma 2.3(1) of [Go]. □

Corollary 4.4. *Suppose* $I^3 = I^2J$. *Then*

(i) *If* $\lambda(I^2/IJ) = 1 = \lambda(\mathfrak{m}I^2/\mathfrak{m}JI)$, *then* $\operatorname{depth}(F(I)) \geq d-1$.
(ii) *If* $\lambda(I^2/IJ) = 2$ *and* $\lambda(\mathfrak{m}I^2/\mathfrak{m}IJ) = 1$, *then* $\operatorname{depth}(F(I)) \geq d-2$.
(iii) *If* $\lambda(I^2/IJ) = 2 = \lambda(\mathfrak{m}I/\mathfrak{m}J)$ *and* $\mathfrak{m}I^2 = \mathfrak{m}IJ$, *then* $\operatorname{depth}(F(I)) \geq d-2$.

Proof. Assume $\lambda(I^2/IJ) = 1 = \lambda(\mathfrak{m}I^2/\mathfrak{m}JI)$ and $I^3 = I^2J$. Then by the Corollary 2.3(a) in [Gu2] we get $\operatorname{depth}(G(I)) \geq d-1$. Hence by the Corollary 3.5 we get $\operatorname{depth}(F(I)) \geq d-1$. This proves (i). Now assume $\lambda(I^2/IJ) = 2$ and $\lambda(\mathfrak{m}I^2/\mathfrak{m}JI) = 1$. Then by the Corollary 2.3(b) in [Gu2] we have $\operatorname{depth}(G(I)) \geq d-2$. Hence by the Corollary 3.5 we get $\operatorname{depth}(F(I)) \geq d-2$. This proves (ii). Now assume $\lambda(I^2/IJ) = 2 = \lambda(\mathfrak{m}I/\mathfrak{m}J)$ and $\mathfrak{m}I^2 = \mathfrak{m}IJ$. Then by the Corollary 2.3(b) in [Gu2] we have $\operatorname{depth}(G(I)) \geq d-2$. Since $\operatorname{depth}(G(I)) \geq d-2 > d-3$, by the Theorem 4.2 we get $\operatorname{depth}(F(I)) \geq d-2$. This proves (iii). □

We conclude this section by characterizing Cohen-Macaulayness of fiber cones ideals with $\sum_{n\geq 1}\lambda(\mathfrak{m}I^n/\mathfrak{m}JI^{n-1}) = 2$.

Proposition 4.5. *Let* $(R,\mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d\geq 2$, I *an* $\mathfrak{m}$*-primary ideal such that* $\operatorname{depth} G(I) \geq d-1$ *and* J *a minimal reduction of* I.

(1) *If* $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1 = \lambda(\mathfrak{m}I^2/\mathfrak{m}IJ)$ *and* $\mathfrak{m}I^{n+1} = \mathfrak{m}I^nJ$ *for all* $n\geq 2$, *then* $F(I)$ *is Cohen-Macaulay if and only if* $\mathfrak{m}I^2 \cap JI = \mathfrak{m}IJ$.
(2) *If* $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 2$ *and* $\mathfrak{m}I^2 = \mathfrak{m}JI$, *then* $F(I)$ *is Cohen-Macaulay.*

Proof. (1) By Proposition 5.4 of [JV2], $F(I)$ is Cohen-Macaulay if and only if $\lambda(\mathfrak{m}I^n + JI^{n-1}/JI^{n-1}) = 1$ for all $n = 1, \ldots, r_J^{\mathfrak{m}}(I)$. Since $r_J^{\mathfrak{m}}(I) = 2$, this equation translates to $\lambda(\mathfrak{m}I^2 + JI/JI) = 1$. Since $\lambda(\mathfrak{m}I^2/\mathfrak{m}JI) = 1$, this is equivalent to $\mathfrak{m}I^2 \cap JI = \mathfrak{m}JI$.

(2) Note that in this case $r_J^{\mathfrak{m}}(I) = 1$. By Corollary 3.2(2), $F(I)$ is Cohen-Macaulay. $\square$

5. Examples

Example 5.1. *(Example 4.6,* [JV2]*) Let* $R = \mathbb{Q}[\![x, y, z]\!]$. *Let* $I = (-x^2 + y^2, -y^2 + z^2, xy, yz, zx)$ *and* $J = (-x^2 + y^2, -y^2 + z^2, xy)$. *Since* $I^3 = JI^2$, J *is a minimal reduction of* I. *Also* $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$ *and* $\mathfrak{m}I^2 = \mathfrak{m}JI$. *Therefore* $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$. *It has been shown in* [JV2] *that* $\operatorname{depth} F(I) = 1$ *and* $\operatorname{depth} G(I) = 0$. *Here the* $\mathfrak{m}$*-reduction number of* I *is 1. This example shows that even if the* $\mathfrak{m}$*-reduction number of* I *is 1, the depth of fiber cone can be quite low without high depth assumption on the associated graded ring.*

Example 5.2. *Let* $R = \mathbb{Q}[\![X,\ Y,\ Z,\ W,\ U]\!]/K$, *where* $K = (-X^3Z + Y^3,\ X^5 - Z^2,\ -X^2Y^3 + Z^3,\ -X^4Y^2 + ZW,\ -X^2Z^2 + YW,\ -Y^2Z + XW,\ XYZ^3 - W^2,\ X^3YW - Z^4,\ Z^5 - Y^4W,\ -Y^5 + X^4W)$. *Then it can be seen that* $R \cong \mathbb{Q}[\![t^6,\ t^{11},\ t^{15},\ t^{31},\ u]\!]$, *is Cohen-Macaulay of dimension* 2. *Let* $x = X + K,\ y = Y + K,\ z = Z + K,\ w = W + K$ *and* $u = U + K$. *Let* $\mathfrak{m} = (x, y, z, w, u)$, $I = (x, y, w, u)$ *and* $J = (x, u)$. *Then it can be seen that* $I^3 = JI$, $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 2$, $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}JI) = 1$, $\mathfrak{m}I^{n+1} \cap J = \mathfrak{m}JI^n$ *for all* $n \geq 2$ *and* $\lambda(\mathfrak{m}I^2/\mathfrak{m}JI) = 1$. *It can also be seen that* $I^2 \cap J = JI$. *Therefore* $I^n \cap J = JI^{n-1}$ *for all* $n \geq 1$. *This implies that* $G(I)$ *is Cohen-Macaulay, i.e.,* $\operatorname{depth} G(I) = 2$. *Since* $\mathfrak{m}I^2 \cap J \neq \mathfrak{m}JI$, $F(I)$ *is not Cohen-Macaulay. It can also be verified that* $\mathfrak{m}I^n \cap (u) \subseteq \mathfrak{m}I^{n-1}J$ *for all* $n \geq 1$ *so that* u^o *is regular in* $F(I)$. *Hence* $\operatorname{depth} F(I) = 1$. *Therefore* $\sum_{n\geq 2} \lambda(\mathfrak{m}I^n \cap J/\mathfrak{m}JI^{n-1}) = 1$ *with* $\operatorname{depth} G(I) = 2$ *and* $\operatorname{depth} F(I) = 1$.

Example 5.3. *Let* $R = \mathbb{Q}[\![x, y, z]\!]$, $I = (x^3, y^3, z^3, xyz, xz^2, yz^2)$ *and* $J = (x^3, y^3, z^3)$. *Then it can be checked that* $r_J(I) = r_J^{\mathfrak{m}}(I) = 2$, $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}JI) = 1$ *and* $\mathfrak{m}I^n \cap J = \mathfrak{m}JI^{n-1}$ *for all* $n \geq 3$. *It can also be verified that* $I^2 \cap J \neq JI$. *Therefore* $G(I)$ *is not Cohen-Macaulay. It can also be seen that the Hilbert series*

$$\begin{aligned} H(G(I), t) &= (1 - t)^{-1} H(G(I/(x^3)), t) \\ &= (1 - t)^{-2} H(G(I/(x^3, y^3)), t) \\ &= \frac{15 + 6t + 6t^2}{(1 - t)^3}. \end{aligned}$$

Therefore $(x^3)^*, (y^3)^*$ *is a regular sequence in* $G(I)$. *Hence* $\operatorname{depth} G(I) = 2$. *Also, it can be computed that* $e_0(F(I)) = 9 = 1+\lambda(I/J+\mathfrak{m}I)+\lambda(I^2/JI+\mathfrak{m}I^2)$. *Therefore by Theorem 2.1 of* [DRV], $F(I)$ *is Cohen-Macaulay.*

Example 5.4. *Let* $R = \mathbb{Q}[\![x, y]\!]$, $I = (x^5, x^3y^2, x^2y^4, y^5)$ *and* $J = (x^5, y^5)$. *Then it can be verified that* $I^5 = JI^4$, $\lambda(\mathfrak{m}I^2 \cap J/\mathfrak{m}JI) = 1$ *and* $\mathfrak{m}I^{n+1} = \mathfrak{m}JI^n$ *for all* $n \geq 2$. *Let* $u = x^5 + y^5$. *Then the Hilbert Series*

$$\begin{aligned} H(G(I), t) &= (1-t)^{-1} H(G(I/u), t) \\ &= \frac{18 + 6t + t^4}{(1-t)^2}. \end{aligned}$$

Therefore u^* *is regular in* $G(I)$. *Since* $I^2 \cap J \neq JI$, *by Valabrega-Valla condition,* $G(I)$ *is not Cohen-Macaulay. Hence* $\operatorname{depth} G(I) = 1$. *Therefore by Theorem 3.3,* $\operatorname{depth} F(I) \geq 1$. *It can also be verified that* $\mu(I^3) = 11 < 13 = \sum_{n=0}^{3}(n+1)\lambda(I^{3-n}/JI^{3-n-1} + \mathfrak{m}I^{3-n})$. *Therefore by Theorem 2.1 of* [DRV], $F(I)$ *is not Cohen-Macaulay. Hence* $\operatorname{depth} F(I) = 1$. *This shows that the lower bound for the depth of* $F(I)$ *given in Theorem 3.3 is sharp.*

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